TOWARDS STONE DUALITY FOR TOPOLOGICAL THEORIES

DIRK HOFMANN AND ISAR STUBBE

ABSTRACT. In the context of categorical topology, more precisely that of $\mathcal{T}$-categories [Hof07], we define the notion of $\mathcal{T}$-colimit as a particular colimit in a V-category. A complete and cocomplete V-category in which limits distribute over $\mathcal{T}$-colimits, is to be thought of as the generalisation of a (co-)frame to this categorical level. We explain some ideas on a $\mathcal{T}$-categorical version of "Stone duality", and show that Cauchy completeness of a $\mathcal{T}$-category is precisely its sobriety.

Introduction

Let X be a topological space, then $\Omega(X)$, its collection of open subsets, is a *frame*: a complete lattice in which finite infima distribute over arbitrary suprema. If $f\colon X \longrightarrow Y$ is a continuous function between topological spaces, then its inverse image $\Omega(f)\colon \Omega(Y) \longrightarrow \Omega(X)$ is a *frame homomorphism*, i.e. a (necessarily order-preserving) function that preserves finite infima and arbitrary suprema. Thus we obtain a contravariant functor, $\Omega\colon \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Frm}$, from the category of topological spaces and continuous functors to that of frames and frame homomorphisms. It is well known that this functor admits a left adjoint

$$\begin{tikzcd} \mathsf{Top}^{\mathrm{op}} \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & \mathsf{Frm} \arrow[l, bend right, "\mathrm{pt}"'] \end{tikzcd}$$

which assigns to any frame F the topological space $\mathrm{pt}(F)$ of its *points*: it is the set $\mathsf{Frm}(F, 2)$ of frame homomorphisms from F to the two-element chain, with open subsets $\{\{p \in \mathrm{pt}(F) \mid p(a) = 1\} \mid a \in F\}$. If the natural continuous comparison $\eta_X\colon X \longrightarrow \mathrm{pt}(\Omega(X))$ is bijective (in which case it actually is a homeomorphism), then X is said to be *sober*. (And because X is T_0 if and only if η_X is injective, we get that a T_0 space is sober if and only if η_X is surjective.) Much more can be said about the interplay between topological spaces and frames; we refer to the classic [Joh86].

Since M. Barr's work [Bar70] we know that topological spaces and continuous functions are precisely the lax algebras and their lax homomorphisms for the lax extension to Rel of the ultrafilter monad on Set. (The algebras for the monad itself are the compact Hausdorff spaces.) With this in mind, in recent years others have studied more generally the lax extension of monads $T\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ to the category $\mathsf{V}\text{-}\mathsf{Mat}$ of matrices with elements in a quantale V [CH03, CT03, Sea05]: the lax algebras, often referred to as (T, V)-categories or (T, V)-algebras in those references, but we shall call them simply $\mathcal{T}$-categories as in [Hof07], are then to be thought of as "topological categories". Examples include, beside topological spaces, also approach spaces, V-enriched categories, metric spaces, multicategories, and more.

Altogether this then raises a natural question: how should we define "$\mathcal{T}$-frames" as the analogue of frames? Is there any hope for a duality between $\mathcal{T}$-categories and "$\mathcal{T}$-frames", generalising that between topological spaces and frames? This is the problem that we address in this paper.

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More exactly, we study the generalisation of the notion of *co-frame* in the context of $\mathcal{T}$-categories. To give an idea of the main difficulty, reconsider the definition: a co-frame is a complete ordered set in which *finite* suprema distribute over (arbitrary) infima. To translate this statement to the context of V-enriched categories, we know that infima and suprema will become enriched limits and colimits, and the distributivity will be expressed by a certain functor being continuous (see e.g. [KS05] for examples in the realm of enriched categories). But how should we translate the *finiteness* of the involved suprema/colimits? This is precisely the point where, besides the categorical data (i.e. the categories enriched in a quantale V), we must make use of the additional topological data (i.e. the monad T on Set): in Definition 2.3 we thus propose the notion of "$\mathcal{T}$-supremum" in a V-category, to be thought of as a "finite supremum", where the *finiteness* relates (perhaps not surprisingly) to the notion of *compactness* relative to the given monad T, as developed in [Hof07]. (More generally, we define "$\mathcal{T}$-colimits" in Definition 2.6; and a V-category is $\mathcal{T}$-cocomplete if and only if it is tensored and has $\mathcal{T}$-suprema, cf. Proposition 2.7.) In Definition 2.9 we then define a "$\mathcal{T}$-frame" to be a complete V-category in which $\mathcal{T}$-suprema suitably distribute over limits. (Note that we speak of $\mathcal{T}$-frames even though we generalise the notion of co-frames.) Of course, these notions are so devised that, when applied to $\mathsf{V} = 2$ (= the two-element chain) and $T = U$ (= the ultrafilter monad), so that $\mathcal{T}$-categories are precisely topological spaces, we eventually recover the ordinary co-frames, as shown in Proposition 3.5 and further on. For the general case, we show in Corollaries 2.10 and 2.11 that there is a pair of functors

$$\begin{tikzcd} \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \arrow[r, bend right, "\Omega"'] & \mathcal{T}\text{-}\mathsf{Frm}. \arrow[l, bend right, "\mathrm{pt}"'] \end{tikzcd}$$

Even though at this point we are unable to prove that these are adjoint, we do show in Proposition 2.12 that there is a natural transformation $\mathrm{Id} \Rightarrow \mathrm{pt} \circ \Omega$; and in Theorem 2.13 we do prove that the $\mathcal{T}$-categories for which the comparison $X \longrightarrow \mathrm{pt}(\Omega(X))$ is surjective, are precisely those which are *Cauchy complete*, which is indeed the expected generalisation of sobriety [Law73, CH09].

We see this work as a first step towards an eventual "Stone duality" for $\mathcal{T}$-categories, and hope that by explaining our ideas, further research on this topic shall be stimulated.

1. The setting: strict topological theories

M. Barr [Bar70] showed in what sense topological spaces can be thought of as algebras: If we write $U\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ for the ultrafilter monad, with multiplication $m\colon U \circ U \Rightarrow U$ and unit $e\colon \mathrm{Id}_{\mathsf{Set}} \Rightarrow U$, then its category of Eilenberg-Moore algebras is precisely that of compact Hausdorff spaces and continuous maps [Man69]. But U admits a *lax extension* to Rel, the quantaloid of sets and relations: define $U'\colon \mathsf{Rel} \longrightarrow \mathsf{Rel}$ to agree with U on the objects, and for a relation $r\colon X \relbar\joinrel\mapsto Y$ with projection maps $p\colon R \longrightarrow X$ and $q\colon R \longrightarrow Y$ put $U'(r) = Uq \cdot (Up)^\circ$. Then U' is still a functor, and the unit and the multiplication of the ultrafilter monad become oplax natural transformations. Hence (U', m, e) is no longer a monad but rather a *lax monad*. Nevertheless, the *lax algebras* for (U', m, e) are precisely topological spaces, and the lax algebra homomorphisms turn out to be exactly the continuous maps.

This situation can be generalised, not only by considering other monads $(T, m, e)\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ besides the ultrafilter monad, but also by studying their lax extensions to quantaloids $\mathsf{V}\text{-}\mathsf{Mat}$ of matrices with elements in a commutative quantale V. (In this paper, $\mathsf{V} = (\mathsf{V}, \bigvee, \otimes, k)$ will always stand for a commutative, unital quantale: $(\mathsf{V}, \bigvee)$ is a complete lattice, in which the supremum of a family $(x_i)_{i \in i}$ is written as $\bigvee_i x_i$, together with an associative and commutative operation $\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}\colon (x, y) \mapsto x \otimes y$ with two-sided unit $k \in \mathsf{V}$, such that both $x \otimes -$ and $- \otimes y$ preserve arbitrary suprema. When one takes

V to be the two-element chain, then it turns out that V-Mat is simply Rel, as we explain further on.) The lax algebras for a lax extension of T to V-Mat are then to be thought of as "topological categories". Of course one has to put conditions on the involved monad and quantale to prove results (in fact, to even define a lax extension and its lax algebras). Over the last decade, several categorical topologists have considered different conditions on T and V [CH04, Sea05, Sea09]; in this paper we shall use, up to a slight rephrasing, the notion of *strict topological theory* as recently put forward in [Hof07].

Definition 1.1. A *strict topological theory* $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ consists of:

(1) a monad $\mathbb{T} = (T, m, e)$ on Set (with multiplication m and unit e),
(2) a commutative quantale $\mathsf{V} = (\mathsf{V}, \bigvee, \otimes, k)$,
(3) a function $\xi\colon T(\mathsf{V}) \longrightarrow \mathsf{V}$,

such that

(a) T sends pullbacks to weak pullbacks and each naturality square of m is a weak pullback (in other words, T and m satisfy the Bénabou-Beck-Chevalley condition),
(b) (V, ξ) is a $\mathbb{T}$-algebra and the monoid structure on V in $(\mathsf{Set}, \times, 1)$ lifts to monoid structure on (V, ξ) in $(\mathsf{Set}^{\mathbb{T}}, \times, 1)$,
(c) writing $P_{\mathsf{V}}\colon \mathsf{Set} \longrightarrow \mathsf{Ord}$ for the functor that sends a function $f\colon X \longrightarrow Y$ to the left adjoint of the "inverse image" $f^{-1}\colon \mathsf{V}^Y \longrightarrow \mathsf{V}^X\colon \varphi \mapsto \varphi \cdot f$ (where V^X is the set of functions from X to V, with pointwise order), the functions $\xi_X\colon V^X \longrightarrow V^{T(X)}\colon f \mapsto \xi \cdot T(f)$ (for X in Set) are the components of a natural transformation $(\xi_X)_X\colon P_{\mathsf{V}} \Rightarrow P_{\mathsf{V}} \circ T$.

Regarding condition (b) in the above definition, note that a quantale V is, in particular, a *set* equipped with *functions* $\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}\colon (x, y) \mapsto x \otimes y$ and $1 \longrightarrow \mathsf{V}\colon * \mapsto k$ (where $1 = \{*\}$ is a generic singleton) satisfying (diagrammatic) associativity and unit axioms; put briefly, $(\mathsf{V}, \otimes, k)$ is a monoid in the cartesian category Set. But now we ask for a function $\xi\colon T(\mathsf{V}) \longrightarrow \mathsf{V}$ making (V, ξ) a $\mathbb{T}$-*algebra*, hence it is natural to require that the functions $(x, y) \mapsto x \otimes y$ and $* \mapsto k$ are in fact $\mathbb{T}$-*homomorphisms*, that is, the following diagrams have to commute:

$$\begin{tikzcd}
T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T(-\otimes-)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] & T(\mathsf{V}) \arrow[d, "\xi"] & T(1) \arrow[l, "T(k)"'] \arrow[d, "!"] \\
\mathsf{V}\times\mathsf{V} \arrow[r, "-\otimes-"'] & \mathsf{V} & 1 \arrow[l, "k"]
\end{tikzcd}$$

Put differently, the monoidal structure $(\mathsf{V}, \otimes, k)$ must lift from Set to the cartesian category $\mathsf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras and homomorphisms. Moreover, it then follows – as shown in [Hof07, Lemma 3.2] – that the *closed* structure on V, in other words, the "internal hom" defined by $x \otimes y \leq z \iff x \leq \hom(y, z)$, then automatically satisfies

$$\begin{tikzcd}
T(\mathsf{V}\times\mathsf{V}) \arrow[r, "T(\hom)"] \arrow[d, "\langle \xi\cdot T\pi_1, \xi\cdot T\pi_2\rangle"'] \arrow[dr, phantom, "\geq"] & T\mathsf{V} \arrow[d, "\xi"] \\
\mathsf{V}\times\mathsf{V} \arrow[r, "\hom"'] & \mathsf{V}
\end{tikzcd}$$

Examples 1.2. The leading examples of strict topological theories are:

(1) the trivial theory: For any quantale V we can consider the theory whose monad-part is the identity monad on Set and for which the required $\xi\colon \mathsf{V} \longrightarrow \mathsf{V}$ is the identity function. We write this trivial strict topological theory as $\mathcal{I}_{\mathsf{V}}$.

(2) the classical ultrafilter theory: Let V be the 2-element chain $\mathsf{2}$ (to be thought of as the "classical truth values"), and consider the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set. Together with the obvious function $\xi\colon U(\mathsf{2}) \longrightarrow \mathsf{2}$ this makes up a strict topological theory which we write as $\mathcal{U}_{\mathsf{2}}$.

(3) the metric ultrafilter theory: Let V be the quantale $([0,\infty], \bigwedge, +, 0)$ of extended non-negative real numbers [Law73], and consider again the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set. Together with the function

$$\xi\colon U([0,\infty]) \longrightarrow [0,\infty], \quad \mathfrak{x} \mapsto \bigwedge\{v \in [0,\infty] \mid [0,v] \in \mathfrak{x}\}$$

this makes up a strict topological theory, written $\mathcal{U}_{[0,\infty]}$.

(4) the general ultrafilter theory: If V is any commutative and integral quantale (meaning that $a \otimes b = b \otimes a$ and that $k = \top$) which is completely distributive, then the ultrafilter monad $\mathbb{U} = (U, m, e)$ on Set together with the function

$$\xi\colon U(\mathsf{V}) \longrightarrow \mathsf{V}\colon \mathfrak{x} \mapsto \bigwedge_{A \in \mathfrak{x}} \bigvee A$$

is a strict topological theory provided that $\otimes\colon \mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}$ is continuous with respect to the compact Hausdorff topology ξ on V. This generalises the two previous examples; details are in [Hof07].

(5) the word theory: For any quantale V, the word monad $\mathbb{L} = (L, m, e)$ on Set together with the function

$$\xi\colon L(\mathsf{V}) \longrightarrow \mathsf{V}\colon (v_1, \dots, v_n) \mapsto v_1 \otimes \dots \otimes v_n\ ,\ (\,) \mapsto k$$

determine a strict topological theory $\mathcal{L}_{\mathsf{V}}$.

In what follows we shall write $\mathsf{V}\text{-}\mathsf{Mat}$ for the quantaloid of V-matrices: its objects are sets, an arrow $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ is a "matrix" whose entries are elements of V, indexed by $Y \times X$. The composition of $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ with $s\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ is $s \cdot r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Z$ whose (z,x)-th element is $\bigvee_{y \in Y} s(z,y) \otimes r(y,x)$; the identity on a set X is the obvious diagonal matrix, with k's on the diagonal and $\bot$'s elsewhere. It is the elementwise supremum of parallel matrices that finally makes $\mathsf{V}\text{-}\mathsf{Mat}$ a quantaloid (in fact, it is the free direct-sum completion of V in the category of quantaloids). As any quantaloid, $\mathsf{V}\text{-}\mathsf{Mat}$ is biclosed (some authors say "left- and right-closed", others say simply "closed"), in the sense that for any matrix $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ and any object Z, both order-preserving functions $- \cdot r\colon \mathsf{V}\text{-}\mathsf{Mat}(Y,Z) \longrightarrow \mathsf{V}\text{-}\mathsf{Mat}(X,Z)$ and $r \cdot -\colon \mathsf{V}\text{-}\mathsf{Mat}(Z,X) \longrightarrow \mathsf{V}\text{-}\mathsf{Mat}(Z,Y)$ admit right adjoints: we shall write

$$s \cdot r \le t \iff s \le t \bullet\!\!- r \quad \text{and} \quad r \cdot p \le q \iff p \le r -\!\!\bullet q$$

for these liftings and extensions. Finally we mention that mapping a matrix $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ to $r^\circ\colon Y \relbar\joinrel\mapstochar\joinrel\rightarrow X$, defined by $r^\circ(x,y) := r(y,x)$, defines an involution $(-)^\circ\colon \mathsf{V}\text{-}\mathsf{Mat}^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Mat}$.

A topological theory $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$ allows for a lax extension of the functor $T\colon \mathsf{Set} \longrightarrow \mathsf{Set}$ to a 2-functor $T_\xi\colon \mathsf{V}\text{-}\mathsf{Mat} \longrightarrow \mathsf{V}\text{-}\mathsf{Mat}$ as follows: we put $T_\xi X = TX$ for each set X, and

$$T_\xi r\colon TY \times TX \longrightarrow \mathsf{V} : (\mathfrak{y}, \mathfrak{x}) \mapsto \bigvee \left\{ \xi \cdot Tr(\mathfrak{w}) \,\middle|\, \mathfrak{w} \in T(Y \times X), T\pi_1(\mathfrak{w}) = \mathfrak{y}, T\pi_2(\mathfrak{w}) = \mathfrak{x} \right\}$$

for each V-matrix $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$. Furthermore, we have $T_\xi(r^\circ) = T_\xi(r)^\circ$ (and we write $T_\xi r^\circ$) for each V-matrix $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$, m becomes a natural transformation $m\colon T_\xi T_\xi \Rightarrow T_\xi$ and e an op-lax natural transformation $e\colon \mathrm{Id} \Rightarrow T_\xi$, i.e. $e_Y \cdot r \le T_\xi r \cdot e_X$ for all $r\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow Y$ in $\mathsf{V}\text{-}\mathsf{Mat}$.

A V-matrix of the form $\alpha\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow TY$ we call *$\mathcal{T}$-matrix* from X to Y, and write $\alpha\colon X \relbar\joinrel\rightharpoonup Y$. For $\mathcal{T}$-matrices $\alpha\colon X \relbar\joinrel\rightharpoonup Y$ and $\beta\colon Y \relbar\joinrel\rightharpoonup Z$ we define as usual the *Kleisli composition*

$$\beta \circ \alpha := m_X \cdot T_\xi \beta \cdot \alpha.$$

This composition is associative and has the $\mathcal{T}$-matrix $e_X\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ as a lax identity: $a \circ e_X \geq a$ and $e_Y \circ a = a$ for any $a\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$.

We now come to the definition of the "topological categories" that we were after in the first place.

Definition 1.3. Let $\mathcal{T}$ be a strict topological theory. A $\mathcal{T}$-*graph* is a pair (X, a) consisting of a set X and a $\mathcal{T}$-matrix $a\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup X$ satisfying $e_X \leq a$. A $\mathcal{T}$-*category* (X, a) is a $\mathcal{T}$-graph such that moreover $a \circ a \leq a$. Given two $\mathcal{T}$-graphs (resp. $\mathcal{T}$-categories) (X, a) and (Y, b), a function $f\colon X \longrightarrow Y$ is a $\mathcal{T}$-*graph morphism* (resp. $\mathcal{T}$-*functor*) if $Tf \cdot a \leq b \cdot f$. Given two $\mathcal{T}$-categories (X, a) and (Y, b), a $\mathcal{T}$-matrix $\varphi\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ is a $\mathcal{T}$-*distributor*, denoted as $\varphi\colon (X, a) \relbar\joinrel\circ\joinrel\rightharpoonup (Y, b)$, if $\varphi \circ a \leq \varphi$ and $b \circ \varphi \leq \varphi$.

Proposition 1.4. *Let $\mathcal{T}$ be a strict topological theory. $\mathcal{T}$-graphs and $\mathcal{T}$-graph morphisms, resp. $\mathcal{T}$-categories and $\mathcal{T}$-functors, form a category $\mathcal{T}$-Gph, resp. $\mathcal{T}$-Cat, for the obvious composition and identities. $\mathcal{T}$-categories and $\mathcal{T}$-distributors between them form a locally ordered category $\mathcal{T}$-Dist, with the Kleisli convolution as composition and the identity on (X, a) given by $a\colon (X, a) \relbar\joinrel\circ\joinrel\rightharpoonup (X, a)$.*

Examples 1.5. We come back to the theories of Example 1.2:

(1) Trivial theory: For each quantale V, $\mathcal{I}_{\mathsf{V}}$-categories are precisely V-categories and $\mathcal{I}_{\mathsf{V}}$-functors are V-functors. As usual, we write V-Cat instead of $\mathcal{I}_{\mathsf{V}}$-Cat, V-Gph instead of $\mathcal{I}_{\mathsf{V}}$-Gph, and so on. In particular, V-Cat is the category Ord of ordered sets if $\mathsf{V} = \mathsf{2}$, and for $\mathsf{V} = [0, \infty]$ one obtains Lawvere's category Met of generalised metric spaces [Law73].
(2) Ultrafilter theories: The main result of [Bar70] states that $\mathcal{U}_{\mathsf{2}}$-Cat is isomorphic to the category Top of topological spaces. In [CH03] it is shown that $\mathcal{U}_{[0,\infty]}$-Cat is isomorphic to the category App of approach spaces [Low97].

Since we always have $\varphi \circ a \geq \varphi$ and $b \circ \varphi \geq \varphi$, the $\mathcal{T}$-distributor condition above implies equality. The local order in $\mathcal{T}$-Dist is inherited from V-Mat, but whereas the latter is a quantaloid (i.e. has local suprema which are stable under composition), the former generally is not. In fact, the matrix-infimum of distributors is a distributor, but the matrix-supremum of distributors is not necessarily a distributor. It is easy to see that all liftings (i.e. right adjoints to $\psi \circ -$) exist in $\mathcal{T}$-Dist, but the example below shows that extensions (i.e. right adjoints to $- \circ \psi$) needn't exist.

Lemma 1.6. *For any $\psi\colon (Y, b) \relbar\joinrel\circ\joinrel\rightharpoonup (X, a)$ and (Z, c) in $\mathcal{T}$-Dist*[1]*, the order-preserving map*

$$\psi \circ -\colon \mathcal{T}\text{-}\mathsf{Dist}((Z, c), (Y, b)) \longrightarrow \mathcal{T}\text{-}\mathsf{Dist}((Z, c), (X, a))$$

admits a right adjoint.

Proof. For any $\gamma\colon (Z, c) \relbar\joinrel\circ\joinrel\rightharpoonup (X, a)$ we pass from

$$\begin{tikzcd} X & Z \arrow[l, "\gamma"', "\circ" marking] \\ Y \arrow[u, "\psi", "\circ" marking] & \end{tikzcd} \qquad \text{to} \qquad \begin{tikzcd} TX & Z \arrow[l, "\gamma"', "+" marking] \\ TTX \arrow[u, "m_X", "+" marking] & \\ TY \arrow[u, "T_{\xi}\psi", "+" marking] & \end{tikzcd}$$

and put $\psi \multimap \gamma := (m_X \cdot T_{\xi}\psi) \mathrel{-\!\!\bullet} \gamma$: it is easily verified that $\psi \multimap \gamma$ is a $\mathcal{T}$-distributor and satisfies the required universal property. □

[1]In fact, this proof also works in the locally ordered category $\mathcal{T}$-URel of so-called *unitary $\mathcal{T}$-relations*: its objects are sets and its arrows are those $a\colon X \relbar\joinrel\mapstochar\joinrel\rightharpoonup Y$ for which $a \circ e_X = a$ holds. Kleisli convolution is composition, and the identity on X is e_X.

Example 1.7. Consider the real numbers with their Euclidian topology, $\mathbb{R}_E$, and with the discrete topology, $\mathbb{R}_D$. Then certainly $f\colon \mathbb{R}_D \longrightarrow \mathbb{R}_E,\ x \mapsto x$ is continuous. Further one checks that a distributor $\theta\colon R_E \multimap E$, resp. $\kappa\colon R_D \multimap E$, is "the same as" a closed subset of $\mathbb{R}$ for the respective topologies, where E denotes a one-element space. Finally, one finds that $\theta \circ f_* = \theta$ for any $\theta\colon R_E \multimap E$. Because the supremum of closed subsets in $\mathbb{R}_E$ is in general different from their supremum in $\mathbb{R}_D$ (i.e. their union), we now find that $- \circ f_*$ does not necessarily preserve such suprema.

We shall now establish the expected relation between $\mathcal{T}$-functors and $\mathcal{T}$-distributors: each $\mathcal{T}$-functor induces an adjoint pair of $\mathcal{T}$-distributors (see [CH09]).

Let $X = (X, a)$ and $Y = (Y, b)$ be $\mathcal{T}$-categories and $f\colon X \longrightarrow Y$ be a $\mathcal{T}$-functor. We define $\mathcal{T}$-distributors $f_*\colon X \multimap Y$ and $f^*\colon Y \multimap X$ by putting $f_* = b \cdot f$ and $f^* = Tf^\circ \cdot b$ respectively. Hence, for $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$, $x \in X$ and $y \in Y$, $f_*(\mathfrak{y}, x) = b(\mathfrak{y}, f(x))$ and $f^*(\mathfrak{x}, y) = b(Tf(\mathfrak{x}), y)$. One easily verifies the rules

$$f^* \circ \varphi = Tf^\circ\varphi \qquad\qquad \text{and} \qquad\qquad \psi \circ f_* = \psi \cdot f,$$

for $\mathcal{T}$-distributors φ and ψ, to conclude that $f_* \dashv f^*$ in $\mathcal{T}$-Dist. One calls a $\mathcal{T}$-category X *Cauchy-complete* (Lawvere complete in [CH09]) if every adjunction $\varphi \dashv \psi$ of $\mathcal{T}$-distributors $\varphi\colon Y \multimap X$ and $\psi\colon X \multimap Y$ is of the form $f_* \dashv f^*$, for some $\mathcal{T}$-functor $f\colon Y \longrightarrow X$. As shown in [CH09], in order to check if X is Cauchy-complete it is enough to consider the case $Y = (1, k_!)$ where $k_! := !^\circ \cdot k\colon 1 \relbar\joinrel\mapstochar\joinrel\rightarrow T1$.

Furthermore, we have functors

$$\mathcal{T}\text{-}\mathsf{Cat} \xrightarrow{(-)_*} \mathcal{T}\text{-}\mathsf{Dist} \xleftarrow{(-)^*} \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}}\ ,$$

where $X_* = X = X^*$ for each $\mathcal{T}$-category $X = (X, a)$. Hence, $\mathcal{T}$-Cat becomes a 2-category via the functor $(-)_*$: we define $f \le g$ if $f_* \le g_*$, which is equivalent to $g^* \le f^*$. Taking this 2-categorical structure into account, the second functor above can be written as $(-)^*\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{coop}} \longrightarrow \mathcal{T}\text{-}\mathsf{Dist}$.

Let us point out some other 2-functors that are of interest (cf. the diagram in figure 1):

- The forgetful $U\colon \mathcal{T}\text{-}\mathsf{Cat} \hookrightarrow \mathcal{T}\text{-}\mathsf{Gph}$ has a left adjoint F which is for instance described in [Hof05].
- Each $\mathcal{T}$-category (X, a) has an underlying V-category $\mathrm{S}(X, a) = (X, e_X^\circ \cdot a)$. This defines a functor $\mathrm{S}\colon \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$ which has a left adjoint $\mathrm{A}\colon \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ defined by $\mathrm{A}(X, r) = (X, T_\xi r \cdot e_X)$.
- As observed in [CH09], there is another functor from $\mathcal{T}$-Cat to V-Cat, namely $\mathrm{M}\colon \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$ which sends a $\mathcal{T}$-category (X, a) to the V-category $(TX, m_X \cdot T_\xi a)$. This functor shall only be needed to define the dual of a $\mathcal{T}$-category (see further) and is not pictured in the diagram.
- Each Eilenberg–Moore $\mathbb{T}$-algebra (X, α) can be considered as a $\mathcal{T}$-category by regarding the function $\alpha\colon TX \longrightarrow X$ as a V-matrix $\alpha^\circ\colon X \relbar\joinrel\mapstochar\joinrel\rightarrow TX$. This defines a functor $\mathrm{D}\colon \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$, whose composition with $\mathsf{Set} \longrightarrow \mathsf{Set}^{\mathbb{T}}$ we denote as $|-|\colon \mathsf{Set} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$.

We shall now discuss some further properties of these functors, especially concerning monoidal structure.

The tensor product $\otimes$ on V has a canonical lifting to V-Mat:one puts $X \otimes Y = X \times Y$, and for V-matrices $a\colon X \longrightarrow Y$ and $b\colon X' \longrightarrow Y'$ one defines

$$(a \otimes b)((x, x'), (y, y')) = a(x, y) \otimes b(y, y').$$

Clearly, any one element set 1 is neutral for this tensor product. Then $T_\xi\colon \mathsf{V}\text{-}\mathsf{Mat} \longrightarrow \mathsf{V}\text{-}\mathsf{Mat}$ together with the natural transformation $m\colon T_\xi T_\xi \Rightarrow T_\xi$ and the op-lax natural transformation $\mathrm{Id} \Rightarrow T_\xi$ becomes a *lax Hopf monad* on $(\mathsf{V}\text{-}\mathsf{Mat}, \otimes, 1)$ in the sense that we have maps $\tau_{X,Y}\colon T(X \times Y) \longrightarrow TX \times TY$ and

$$\begin{tikzcd}
 & \mathcal{T}\text{-}\mathsf{Gph} \arrow[dr, shift left, "F"] & & & \\
\mathsf{V}\text{-}\mathsf{Cat} \arrow[rr, shift left, "A"] & & \mathcal{T}\text{-}\mathsf{Cat} \arrow[ll, shift left, "S"] \arrow[ul, shift left, "U"] \arrow[rr, "(-)_*"] & & \mathsf{Map}(\mathcal{T}\text{-}\mathsf{Dist}) \\
 & \mathsf{Set} \arrow[ur, "|-|"] \arrow[rr, shift left] & & \mathsf{Set}^{\mathbb{T}} \arrow[ll, shift left] \arrow[ul, "D"'] &
\end{tikzcd}$$

FIGURE 1. Some (2-)functors of interest

$!\colon T1 \longrightarrow 1$ so that the diagrams

$$\begin{tikzcd}
T(X\otimes Y) \arrow[r, "\tau_{X,Y}"] \arrow[d, "T_\xi(r\otimes s)"', "\circ" marking] & TX\otimes TY \arrow[d, "T_\xi r\otimes T_\xi s", "\circ" marking] \\
T(X'\otimes Y') \arrow[r, "\tau_{X',Y'}"'] & TX'\otimes TY'
\end{tikzcd}
\qquad\text{and}\qquad
\begin{tikzcd}
T1 \arrow[r, "!"] \arrow[d, "T_\xi k"', "\circ" marking] & 1 \arrow[d, "k", "\circ" marking] \\
T1 \arrow[r, "!"'] & 1
\end{tikzcd}$$

commute in V-Mat. Hence, we cannot speak of a Hopf monad (see [Moe02]) only because (T_ξ, m, e) is just a lax monad on V-Mat. This additional structure permits us to turn also $\mathcal{T}$-Cat into a tensored category: for $\mathcal{T}$-categories (or, more general, $\mathcal{T}$-graphs) $X = (X, a)$ and $Y = (Y, b)$ we define $X \otimes Y = (X \times Y, c)$ where $c = \tau^\circ_{X,Y} \cdot (a \otimes b)$. Explicitly, for $\mathfrak{w} \in T(X \times Y)$, $x \in X$, $y \in Y$, $\mathfrak{x} = T\pi_1(\mathfrak{w})$ and $\mathfrak{y} = T\pi_2(\mathfrak{w})$ we have

$$c(\mathfrak{w}, (x, y)) = a(\mathfrak{x}, x) \otimes b(\mathfrak{y}, y).$$

The $\mathcal{T}$-category $E = (1, k_!)$, with $k_!\colon\; = !^\circ \cdot k\colon 1 {\longrightarrow\!\!\!\!\!\!\mapsto} T1$, is neutral for $\otimes$. This tensor product and its properties are studied in [Hof07] (unfortunately, without mentioning the concept of a Hopf monad). The functors introduced above have now the following properties: $\mathrm{A}\colon \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ and $\mathrm{M}\colon \mathsf{V}\text{-}\mathsf{Cat} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ are op-monoidal 2-functors, $\mathrm{S}\colon \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$ is a strong monoidal 2-functor and $\mathrm{D}\colon \mathsf{Set}^{\mathbb{T}} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$ is a strong monoidal functor.

Another important feature of a topological theory is that it allows us to consider V as a $\mathcal{T}$-category $\mathsf{V} = (\mathsf{V}, \hom_\xi)$, where

$$\hom_\xi\colon T\mathsf{V} \times \mathsf{V} \longrightarrow \mathsf{V}, \quad (\mathfrak{v}, v) \mapsto \hom(\xi(\mathfrak{v}), v).$$

Note that $\mathsf{V} = (\mathsf{V}, \hom_\xi)$ is in general not isomorphic to $\mathrm{A}(\mathsf{V}, \hom)$. Furthermore, V is a monoid in $(\mathcal{T}\text{-}\mathsf{Cat}, \otimes, E)$ since both $k\colon E \longrightarrow \mathsf{V}$ and $\otimes\colon \mathsf{V} \otimes \mathsf{V} \longrightarrow \mathsf{V}$ are $\mathcal{T}$-functors. Through the strong monoidal 2-functor $\mathrm{S}\colon \mathcal{T}\text{-}\mathsf{Cat} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$, this specialises to the usual monoid structure on V in V-Cat. We also remark that $\xi\colon |\mathsf{V}| \longrightarrow \mathsf{V}$ becomes now a $\mathcal{T}$-functor.

We finish this section by presenting a characterisation of $\mathcal{T}$-distributors as V-valued $\mathcal{T}$-functors, generalising therefore a well-known fact about V-categories. This result involves the dual category, a concept which has no obvious $\mathcal{T}$-counterpart. However, the following definition (see [CH09]) proved to be useful: given a $\mathcal{T}$-category $X = (X, a)$, one puts

$$X^{\mathrm{op}} = \mathrm{A}(\mathrm{M}(X)^{\mathrm{op}}).$$

Theorem 1.8 ([CH09])**.** *For $\mathcal{T}$-categories (X, a) and (Y, b), and a $\mathcal{T}$-matrix $\psi\colon X \longrightarrow\!\!\!\!\!\!\mapsto Y$, the following assertions are equivalent.*

(i) $\psi\colon (X, a) \circ\!\!\!\longrightarrow (Y, b)$ *is a $\mathcal{T}$-distributor.*

(ii) *Both* $\psi\colon |Y|\otimes X \longrightarrow \mathsf{V}$ *and* $\psi\colon Y^{\mathrm{op}}\otimes X \longrightarrow \mathsf{V}$ *are* $\mathcal{T}$*-functors.*

In particular, since $a\colon X \,{-\!\!\circ\!\!\rightharpoonup}\, X$ for each $\mathcal{T}$-category $X = (X, a)$, we have two $\mathcal{T}$-functors

$$a\colon |X|\otimes X \longrightarrow \mathsf{V} \qquad\text{and}\qquad a\colon X^{\mathrm{op}}\otimes X \longrightarrow \mathsf{V}.$$

The theorem above, together with the condition $T1 = 1$, can now be used to construct the Cauchy-completion $\mathrm{y}\colon X \longrightarrow \tilde{X}$ of a $\mathcal{T}$-category X, where $\tilde{X}$ has as objects

$$\{\psi \in \mathcal{T}\text{-}\mathsf{Dist}(E, X) \mid \psi \text{ is left adjoint}\}$$

and y is the Yoneda embedding $x \mapsto x_*$. For details we refer to [HT10].

Examples 1.9. We consider first $\mathsf{V} = [0,\infty]$, hence V-category means (generalised) metric space. In [Law73] F.W. Lawvere has shown that equivalence classes of Cauchy sequences in a metric space X correspond precisely to left adjoint $[0,\infty]$-distributors $\psi\colon E \,{-\!\!\circ\!\!\rightharpoonup}\, X$, and a Cauchy sequence converges to x if and only if x is a colimit of the corresponding $[0,\infty]$-distributor. Hence, Cauchy completeness has the usual meaning and $\tilde{X}$ describes the usual Cauchy completion of a metric space.

In [HT10] it is shown that the topological space (= $\mathcal{U}_2$-category) $\tilde{X}$ is homeomorphic to the space of all completely prime filters on the lattice τ of open subsets of X, and $\mathrm{y}\colon X \longrightarrow \tilde{X}$ corresponds to the map which sends $x \in X$ to its neighbourhood filter. Of course, one can equivalently consider right adjoint $\mathcal{U}_2$-distributors $\varphi\colon X \,{-\!\!\circ\!\!\rightharpoonup}\, E$, and in [CH09] it is shown that a $\mathcal{U}_2$-distributors $\varphi\colon X \,{-\!\!\circ\!\!\rightharpoonup}\, E$ is right adjoint if and only if $\varphi\colon X \longrightarrow \mathsf{2}$ is the characteristic map of an irreducible closed subset A of X, and $\varphi = x^*$ if and only $A = \overline{\{x\}}$. Hence, a topological space X is Cauchy complete if and only if X is weakly sober.

2. $\mathcal{T}$-categories versus $\mathcal{T}$-frames

Recall from the Introduction that $\Omega\colon \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Frm}$ is the functor that sends any topological space X to the frame $\Omega(X)$ of its open subsets, and any continuous function $f\colon X \longrightarrow Y$ to the frame homomorphism $\Omega(f)\colon \Omega(Y) \longrightarrow \Omega(X)$ given by inverse image. It is straightforward to see that $\Omega(X)$ is isomorphic (qua ordered set) to $\mathsf{Top}(X, S)$, where S is the Sierpinski space, topological spaces are considered with their specialisation order (which continuous functions preserve), and $\mathsf{Top}(X, S)$ is ordered pointwise. In fact, modulo these isomorphisms, $\Omega\colon \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Frm}$ is simply a corestriction of the representable functor $\mathsf{Top}(-, S)\colon \mathsf{Top}^{\mathrm{op}} \longrightarrow \mathsf{Ord}$. Further recall that the left adjoint to Ω, $\mathrm{pt}\colon \mathsf{Frm} \longrightarrow \mathsf{Top}^{\mathrm{op}}$, is also defined by means of a representable, namely $\mathsf{Frm}(-, \mathsf{2})$. It is now noteworthy that the specialisation order of the Sierpinski space S is precisely the two-element chain $\mathsf{2} = \{0 \le 1\}$, and conversely S is the Alexandrov topology on $\mathsf{2}$. For this reason, some have called the two-point set $\{0, 1\}$ a *dualising object* in this situation: it can be endowed with two different structures, the Sierpinski topology and the total order, making it objects of two different categories, topological spaces and frames, and represents a duality between these categories [PT91].

This analysis now suggests our method to define the category of "$\mathcal{T}$-frames" in the general context of $\mathcal{T}$-categories, as follows. For any strict topological theory $\mathcal{T} = (\mathbb{T}, \mathsf{V}, \xi)$, the quantale V naturally bears the structure of a $\mathcal{T}$-category; thus we have the representable functor $\mathcal{T}\text{-}\mathsf{Cat}(-, \mathsf{V})\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$. Now we devise a category $\mathcal{T}\text{-}\mathsf{Frm}$ of "$\mathcal{T}$-frames" and "$\mathcal{T}$-frame homomorphisms" in such a way[2] that (i) the representable $\mathcal{T}\text{-}\mathsf{Cat}(-, \mathsf{V})\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Cat}$ corestricts to a functor $\Omega\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Frm}$, and (ii) V is an object of $\mathcal{T}\text{-}\mathsf{Frm}$ representing a functor $\mathrm{pt}\colon \mathcal{T}\text{-}\mathsf{Frm} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}}$. Ideally, these functors should then be adjoint, but for now we are unable to prove this. However, we do show a natural comparison $\eta_X\colon X \longrightarrow$

[2] At this point we should mention that, specialised to the topological case, $\mathsf{V} = \mathsf{2}$ becomes the Sierpinski space with $\{1\}$ closed so that $\mathsf{Top}(X, \mathsf{2})$ is naturally isomorphic to the *co-frame* of closed subsets of X. Nevertheless, we prefer to use the term "$\mathcal{T}$-frame" in the sequel.

$\mathrm{pt}(\Omega(X))$ for any $\mathcal{T}$-category X, and we prove that X is a *Cauchy complete* $\mathcal{T}$-category (amounting to *sobriety* in the case of T_0 topological spaces) if and only if η_X is surjective.

For technical reasons **we shall from now on assume that** $T1 = 1$. Together with Theorem 1.8 this implies that a $\mathcal{T}$-distributor $\varphi\colon X \,-\!\!\circ\!\!\rightharpoonup E$ is "the same thing as" a $\mathcal{T}$-functor $\varphi\colon X \longrightarrow \mathsf{V}$ (recall that E is the unit for the tensor product in $\mathcal{T}$-Cat). Furthermore, for any $\alpha\colon X \,-\!\!\!\rightharpoonup E$,

$$-\circ\alpha\colon \mathcal{T}\text{-}\mathsf{Mat}(E,E) \longrightarrow \mathcal{T}\text{-}\mathsf{Mat}(X,E)$$

has a right adjoint $(-) \multimapdotinv \alpha$ calculated as in V-Mat, due to $T_{\xi} v = v$ (for any $v\colon 1 \,-\!\!\!\rightharpoonup 1$). Unfortunately, the condition $T1 = 1$ excludes Example 1.2 (5).

Lemma 2.1. *The following assertions hold.*

(1) $\bigwedge\colon \mathsf{V}^I \longrightarrow \mathsf{V}$ *is a* $\mathcal{T}$*-functor, for each index set* I.
(2) $\hom(v,-)\colon \mathsf{V} \longrightarrow \mathsf{V}$ *is a* $\mathcal{T}$*-functor, for each* $v \in \mathsf{V}$.
(3) $v \otimes -\colon \mathsf{V} \longrightarrow \mathsf{V}$ *is a* $\mathcal{T}$*-functor, for each* $v \in \mathsf{V}$.

It is now straightforward that the representable functor $\mathcal{T}\text{-}\mathsf{Cat}(-,\mathsf{V})\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{Ord}$ lifts to a functor $\mathsf{V}^-\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Cont}$ by putting V^X to be the full sub-V-category of $P(SX)$ (i.e. the usual V-category of covariant V-presheaves on SX, the "specialisation" V-category underlying the $\mathcal{T}$-category X) determined by the elements in $\mathcal{T}\text{-}\mathsf{Cat}(X,\mathsf{V})$. Clearly, a $\mathcal{T}$-functor $f\colon X \longrightarrow Y$ (with Y being a $\mathcal{T}$-category) induces a V-functor $\mathsf{V}^f\colon \mathsf{V}^Y \longrightarrow \mathsf{V}^X$ which preserves infima, tensors and cotensors, i.e. all weighted limits. Being complete, V^Y is also cocomplete, but suprema are typically not computed pointwise and hence in general not preserved by V^f. However, a particular class of suprema are preserved by V^f, as we show next.

Proposition 2.2 ([Hof07]). *Let X be a $\mathcal{T}$-category. Then X is compact if and only if $\bigvee\colon \mathsf{V}^X \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-graph morphism. In particular, $\bigvee\colon \mathsf{V}^X \longrightarrow \mathsf{V}$ is a $\mathcal{T}$-functor for each $\mathbb{T}$-algebra X.*

Here a $\mathcal{T}$-category $X = (X,a)$ is called compact if $k \le \bigvee\{a(\mathfrak{x},x) \mid x \in X\}$, for every $\mathfrak{x} \in UX$. For topological spaces, compact has the usual meaning, and an approach space is compact if and only if its measure of compactness is 0 (see [Low97]). Also note that every V-category is compact. In the proposition above, V^X is the $\mathcal{T}$-graph with structure matrix $[\![-,-]\!]$ defined as

$$[\![\mathfrak{p},\varphi]\!] = \bigwedge_{\substack{\mathfrak{q}\in T(X\times Y^X),x\in X\\ \mathfrak{q}\mapsto\mathfrak{p}}} \hom(a(T\pi_1(\mathfrak{q}),x),\hom(\xi\cdot T\mathrm{ev}(\mathfrak{q}),\varphi(x))).$$

In fact, we apply here to V the right adjoint $(-)^X$ of $X\otimes -\colon \mathcal{T}\text{-}\mathsf{Gph} \longrightarrow \mathcal{T}\text{-}\mathsf{Gph}$ (see [Hof07]). Note that we use here the same notation V^X for the $\mathcal{T}$-graph and the V-category (defined on the same set of objects). However, if $\mathfrak{p} = e_{\mathsf{V}^X}(\varphi')$ with $\varphi' \in \mathsf{V}^X$ in the formula above, then

$$[\![e_{V^X}(\varphi'),\varphi]\!] = \bigwedge_{x\in X} \hom(\varphi'(x),\varphi(x)) = \varphi \multimapdotinv \varphi' = [\varphi',\varphi],$$

i.e. the underlying V-graph of the $\mathcal{T}$-graph V^X is actually the V-category V^X described above. As a consequence, if $\varphi'\colon X \,-\!\!\circ\!\!\rightharpoonup E$ has a left adjoint $\psi\colon E \,-\!\!\circ\!\!\rightharpoonup X$ in $\mathcal{T}$-Dist, then

$$[\![e_{Y^X}(\varphi'),\varphi]\!] = \varphi\circ\psi.$$

Proposition 2.2 suggests now the following new notions.

Definition 2.3. Let $A = (A,a)$ and $B = (B,b)$ be $\mathcal{T}$-graphs whose underlying V-graphs are V-categories; for shorthand, we will write A (resp. B) for both the $\mathcal{T}$-graph and the underlying V-category. A V-functor $f\colon A \longrightarrow B$ is said to be $\mathcal{T}$*-compatible* if, for each $\mathbb{T}$-algebra I and each $\mathcal{T}$-graph morphism $h\colon I \longrightarrow A$,

the composite $f \cdot h$ is a $\mathcal{T}$-graph morphism as well. By a $\mathcal{T}$-*diagram* in A we mean a $\mathcal{T}$-graph morphism $D\colon I \longrightarrow A$ where I is a $\mathbb{T}$-algebra; and a supremum of a $\mathcal{T}$-diagram is a $\mathcal{T}$-*supremum*. Finally, we say that a $\mathcal{T}$-compatible V-functor $\Phi\colon A \longrightarrow B$ *preserves* $\mathcal{T}$-*suprema* if Φ preserves suprema of $\mathcal{T}$-diagrams.

Proposition 2.4. *For every $\mathcal{T}$-functor $f\colon X \longrightarrow Y$, the V-functor $\mathsf{V}^f\colon \mathsf{V}^Y \longrightarrow \mathsf{V}^X$ underlies a $\mathcal{T}$-graph morphism, and hence is $\mathcal{T}$-compatible. Moreover, V^f preserves $\mathcal{T}$-suprema (but in general not all suprema).*

Remark 2.5. We consider the Yoneda morphism $\mathrm{y}\colon X \longrightarrow \tilde{X}$. Then $\mathsf{V}^{\mathrm{y}}\colon \mathsf{V}^{\tilde{X}} \longrightarrow \mathsf{V}^X$ is an isomorphism of V-categories, where V^{y} sends $\tilde{\varphi}\colon \tilde{X} \longrightarrow \mathsf{V}$ to its restriction $\varphi\colon X \longrightarrow \mathsf{V}$. In fact, when considering $\varphi, \tilde{\varphi}$ as $\mathcal{T}$-distributors $\varphi\colon E \,{-\!\!\circ\!\!\rightharpoonup}\, X$ and $\tilde{\varphi}\colon E \,{-\!\!\circ\!\!\rightharpoonup}\, \tilde{X}$, we have $\tilde{\varphi} = \mathrm{y}_* \circ \varphi$ resp. $\varphi = \mathrm{y}^* \circ \tilde{\varphi}$. Hence, since $\mathrm{y}_* \dashv \mathrm{y}^*$ is an equivalence of $\mathcal{T}$-distributors,

$$\tilde{\varphi} \multimapinv \tilde{\psi} = \varphi \multimapinv \psi$$

for all $\tilde{\varphi}, \tilde{\psi}\colon \tilde{X} \longrightarrow \mathsf{V}$. Its inverse $\Phi\colon \mathsf{V}^X \longrightarrow \mathsf{V}^{\tilde{X}}$, $\varphi \longrightarrow \tilde{\varphi}$ certainly preserves all suprema. Moreover, Φ is $\mathcal{T}$-compatible. To see this, let $h\colon I \longrightarrow \mathsf{V}^X$ be a $\mathcal{T}$-graph morphism where I is a $\mathcal{T}$-algebra. Since V is injective with respect to fully faithful $\mathcal{T}$-functors, we have an (in fact unique) extension $l\colon \tilde{X} \otimes I \longrightarrow \mathsf{V}$ of $\llcorner h \lrcorner\colon X \otimes I \longrightarrow \mathsf{V}$ along $\mathrm{y} \otimes \mathrm{id}_I\colon X \otimes I \longrightarrow \tilde{X} \otimes I$. Then $\ulcorner l \urcorner(i) \cdot \mathrm{y} = h(i)$ for each $i \in I$, and therefore $\ulcorner l \urcorner = \Phi \cdot h$.

Definition 2.6. Assume that the underlying V-category of A has all tensors. A $\mathcal{T}$-*weighted* diagram in A is given by a set I together with a $\mathbb{T}$-algebra structure $\alpha\colon TI \longrightarrow I$ and a V-category structure $r\colon I \,{-\!\!\!\mapsto}\, I$, a V-functor $h\colon I \longrightarrow A$ and a V-distributor $\psi\colon 1 {-\!\!\circ\!\!\rightarrow} I$ such that the map

$$I \longrightarrow A,\ i \mapsto \psi(i) \otimes h(i)$$

is a $\mathcal{T}$-graph morphism. The colimit of a $\mathcal{T}$-weighted diagram in A is called a $\mathcal{T}$-*colimit*, and A is $\mathcal{T}$-*cocomplete* if all $\mathcal{T}$-colimits exist in A. A V-distributor $\varphi\colon 1 {-\!\!\circ\!\!\rightarrow} A$ is called $\mathcal{T}$-*generated* if $\varphi = h_* \cdot \psi$ in V-Dist, for h and ψ as above.

Proposition 2.7. *The following assertions are equivalent, for a $\mathcal{T}$-graph $A = (A, a)$ where SA is a V-category.*

(i) *A is $\mathcal{T}$-cocomplete.*
(ii) *A has all tensors and all $\mathcal{T}$-suprema.*
(iii) *Each $\mathcal{T}$-generated V-distributor $\varphi\colon 1 {-\!\!\circ\!\!\rightarrow} A \in P(A)$ has a supremum $\sup_A(\varphi)$ in A.*

Note that a $\mathcal{T}$-compatible and tensor-preserving V-functor $f\colon A \longrightarrow B$ sends $\mathcal{T}$-weighted diagrams in A to $\mathcal{T}$-weighted diagrams in B. In fact, we have

Proposition 2.8. *Let A and B be $\mathcal{T}$-graphs whose underlying V-categories are $\mathcal{T}$-cocomplete, and let $f\colon A \longrightarrow B$ be a $\mathcal{T}$-compatible V-functor which preserves tensors. Then f preserves $\mathcal{T}$-colimits if and only if f preserves $\mathcal{T}$-suprema.*

Based on the considerations above, we now propose a $\mathcal{T}$-equivalent for the concept of a co-frame (but note that we call these "$\mathcal{T}$-frames" and not "$\mathcal{T}$-co-frames"):

Definition 2.9. $\mathcal{T}$-Frm is the locally ordered category with:

objects: $\mathcal{T}$-*frames*, i.e. $\mathcal{T}$-graphs A whose underlying V-graph is a complete V-category satisfying the following *distributivity law*: for any distributor $\varphi\colon I {-\!\!\circ\!\!\rightarrow} 1$ and functor $h\colon I \longrightarrow PA$ such that $h(i)$ is $\mathcal{T}$-generated for all $i \in I$, if $\lim(\varphi, h)$ is $\mathcal{T}$-generated then $\sup_A(\lim(\varphi, h)) = \lim(\varphi, \sup_A \cdot\, h)$.

morphisms: $\mathcal{T}$-*frame homomorphisms*, i.e. $\mathcal{T}$-compatible V-functors between the underlying V-categories of $\mathcal{T}$-frames, that furthermore preserve weighted limits and $\mathcal{T}$-weighted colimits.

By construction, we have a canonical forgetful functor $\mathcal{T}\text{-}\mathsf{Frm} \longrightarrow \mathsf{V}\text{-}\mathsf{Cont}$. Earlier we already explained that $\mathsf{V} \in \mathcal{T}\text{-}\mathsf{Cat}$ and that the representable functor $\mathcal{T}\text{-}\mathsf{Cat}(-,\mathsf{V})\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{Ord}$ lifts to a functor $\mathsf{V}^{-}\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Cont}$. Now we can prove:

Corollary 2.10. *The functor* $\mathsf{V}^{-}\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathsf{V}\text{-}\mathsf{Cont}$ *factors through the forgetful functor* $\mathcal{T}\text{-}\mathsf{Frm} \longrightarrow \mathsf{V}\text{-}\mathsf{Cont}$*; we call the resulting functor* $\Omega\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Frm}$.

Proof. For each $\mathcal{T}$-functor $f\colon X \longrightarrow Y$, the underlying V-graph of the $\mathcal{T}$-graph V^X is a complete V-category, and $\mathsf{V}^f\colon \mathsf{V}^Y \longrightarrow \mathsf{V}^X$ is a $\mathcal{T}$-graph morphism which preserves all weighted limits and all $\mathcal{T}$-weighted colimits. Furthermore $A = \mathsf{V}^X$ satisfies the distributivity axiom in Definition 2.9 since the presheaf V-category $P(SX)$ is completely distributive, and A is closed in $P(SX)$ under weighted limits and $\mathcal{T}$-weighted colimits. $\square$

Since $\mathsf{V} \in \mathcal{T}\text{-}\mathsf{Frm}$, we certainly have a representable functor $\mathcal{T}\text{-}\mathsf{Frm}(-,\mathsf{V})\colon (\mathcal{T}\text{-}\mathsf{Frm})^{\mathrm{op}} \longrightarrow \mathsf{Ord}$. But there is more:

Corollary 2.11. *The functor* $\mathcal{T}\text{-}\mathsf{Frm}(-,\mathsf{V})\colon (\mathcal{T}\text{-}\mathsf{Frm})^{\mathrm{op}} \longrightarrow \mathsf{Ord}$ *lifts to a functor* $\mathrm{pt}\colon (\mathcal{T}\text{-}\mathsf{Frm})^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$.

Proof. This is done by putting on $\mathcal{T}\text{-}\mathsf{Frm}(X,\mathsf{V})$ the largest $\mathcal{T}$-category structure that makes all evaluation maps $\mathrm{ev}_{X,x}\colon \mathcal{T}\text{-}\mathsf{Frm}(X,\mathsf{V}) \longrightarrow \mathsf{V},\ h \mapsto h(x)$ into $\mathcal{T}$-functors. $\square$

Note how, in the two previous corollaries, V plays the role of a *dualising object*: it is on the one hand an object of $\mathcal{T}\text{-}\mathsf{Cat}$, and as such represents the functor $\Omega\colon \mathcal{T}\text{-}\mathsf{Cat}^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Frm}$; but it is also an object of $\mathcal{T}\text{-}\mathsf{Frm}$, and as such represents the functor $\mathrm{pt}\colon (\mathcal{T}\text{-}\mathsf{Frm})^{\mathrm{op}} \longrightarrow \mathcal{T}\text{-}\mathsf{Cat}$. Next we observe:

Proposition 2.12. *There is a natural transformation* $\eta\colon \mathrm{Id} \Rightarrow \mathrm{pt}\cdot\Omega$ *with components*

$$\eta_X\colon X \longrightarrow \mathrm{pt}(\Omega(X)),\ x \mapsto \mathrm{ev}_{X,x} \quad \textit{for} \quad X \in \mathcal{T}\text{-}\mathsf{Cat}.$$

We do not know whether η_X is always fully faithful, but we do have the following result (recall that E is the unit for the tensor in $\mathcal{T}\text{-}\mathsf{Cat}$):

Theorem 2.13. *For any* $X \in \mathcal{T}\text{-}\mathsf{Cat}$, $\mathrm{pt}(\Omega(X))$ *has the same objects as the Cauchy completion* $\tilde{X}$ *of* X. *In fact, we have an isomorphism* $\mathsf{Map}(\mathcal{T}\text{-}\mathsf{Dist})(E,X) \longrightarrow \mathcal{T}\text{-}\mathsf{Frm}(\Omega(X),\mathsf{V})$ *of ordered sets, making the diagram*

$$\begin{tikzcd} & X \arrow[dl, "(-)_*"'] \arrow[dr, "\eta_X"] & \\ \mathsf{Map}(\mathcal{T}\text{-}\mathsf{Dist})(E,X) \arrow[rr] & & \mathcal{T}\text{-}\mathsf{Frm}(\Omega(X),\mathsf{V}) \end{tikzcd}$$

commute. Hence X is Cauchy complete if and only if η_X is surjective.

The proof of the theorem above is the combination of the results below.

Lemma 2.14. *Let* $X = (X,a)$ *be a* $\mathcal{T}$-*category and* $\varphi\colon X \longrightarrow \mathsf{V}$ *be a* $\mathcal{T}$-*functor. Then the representable* V-*functor* $\Phi = [\varphi,-]\colon \Omega(X) \longrightarrow \mathsf{V}$ *is also a* $\mathcal{T}$-*graph morphism and preserves infima and cotensors. Moreover, if* $\psi \dashv \varphi$ *in* $\mathcal{T}\text{-}\mathsf{Dist}$, *then* Φ *preserves also tensors and* $\mathcal{T}$-*suprema.*

Proof. Being a representable V-functor, Φ preserves infima and cotensors. To see that Φ is a $\mathcal{T}$-graph morphism, recall first that

$$[\varphi,\varphi'] = \bigwedge_{x\in X} \hom(\varphi(x),\varphi'(x)).$$

Since $\bigwedge\colon \mathsf{V}^{X_D} \longrightarrow \mathsf{V}$ (with $X_D = (X, e_X)$ being the discrete $\mathcal{T}$-category) is a $\mathcal{T}$-graph morphism, it is enough to show that

$$\Psi\colon \mathsf{V}^X \longrightarrow \mathsf{V}^{X_D},\ \varphi' \mapsto \hom(\varphi(-),\varphi'(-))$$

is a $\mathcal{T}$-graph morphism. But Ψ is just the mate of the composite

$$X_D\otimes\mathsf{V}^X \xrightarrow{\Delta\otimes\mathrm{id}} X_D\otimes X\otimes\mathsf{V}^X \xrightarrow{\varphi\otimes\mathrm{ev}} \mathsf{V}_D\otimes\mathsf{V} \xrightarrow{\hom} \mathsf{V}$$

of $\mathcal{T}$-graph morphisms.

Assume now $\psi \dashv \varphi$ in $\mathcal{T}$-Dist. Then, for any $\varphi'\colon X \longrightarrow \mathsf{V}$ and $v\in\mathsf{V}$,

$$[\varphi, v\otimes\varphi'] = (v\otimes\varphi')\circ\psi = (v\circ\varphi')\circ\psi = v\circ(\varphi'\circ\psi) = v\otimes[\varphi,\varphi'].$$

Finally, to see that $[\varphi,-]$ preserves $\mathcal{T}$-suprema, we assume X to be Cauchy complete. Let $D\colon I \longrightarrow \mathsf{V}^X$, $i\mapsto\varphi_i$ be a $\mathcal{T}$-diagram. Then, since $\varphi = a(e_X(x),-)$ for some $x\in X$,

$$[\varphi,\bigvee_{i\in I}\varphi_i] = [a(e_X(x),-),\bigvee_{i\in I}\varphi_i] = \left(\bigvee_{i\in I}\varphi_i\right)(x) = \bigvee_{i\in I}\varphi_i(x) = \bigvee_{i\in I}[\varphi,\varphi_i]. \qquad\square$$

Hence $\psi \mapsto [\varphi,-]$ where $\psi\dashv\varphi$ defines a map $\mathsf{Map}(\mathcal{T}\text{-}\mathsf{Dist})(E,X) \longrightarrow \mathcal{T}\text{-}\mathsf{Frm}(\Omega(X),\mathsf{V})$, which is clearly injective and hence, by definition, an order-embedding. Before stating our next result, we recall that $\varphi = \bigvee_{\mathfrak{x}\in TX}(a(\mathfrak{x},-)\otimes\xi\cdot T\varphi(\mathfrak{x}))$ for each $\mathcal{T}$-functor $\varphi\colon X\longrightarrow\mathsf{V}$.

Proposition 2.15. *Let $X=(X,a)$ be a $\mathcal{T}$-category and $\Phi\colon\Omega(X)\longrightarrow\mathsf{V}$ be a V-functor. Then the following assertions are equivalent.*

(i) $\Phi = [\varphi,-]$ *for some right adjoint $\mathcal{T}$-distributor $\varphi\colon X \,{\circ\!\!\rightharpoondown}\, E$.*
(ii) Φ *preserves infima, tensors, cotensors and $\mathcal{T}$-suprema.*
(iii) Φ *preserves infima, tensors, cotensors and, for each $\varphi\in\mathsf{V}^X$,*

(*) $$\Phi(\varphi) = \bigvee_{\mathfrak{x}\in TX}\Phi(a(\mathfrak{x},-)\otimes\xi\cdot T\varphi(\mathfrak{x})).$$

Proof. (i)⇒(ii): Follows from the lemma above.
(ii)⇒(iii): It is enough to observe that

$$|X|\otimes X\longrightarrow\mathsf{V},\ (\mathfrak{x},x)\mapsto a(\mathfrak{x},x)\otimes\xi\cdot T\varphi(\mathfrak{x})$$

is a $\mathcal{T}$-functor since it can be written as the composite

$$|X|\otimes X \xrightarrow{\Delta\otimes\mathrm{id}_X} |X|\otimes|X|\otimes X \xrightarrow{T\varphi\otimes a} |\mathsf{V}|\otimes\mathsf{V} \xrightarrow{\xi\otimes\mathrm{id}_X} \mathsf{V}\otimes\mathsf{V} \xrightarrow{\otimes} \mathsf{V}$$

of $\mathcal{T}$-functors.
(iii)⇒(i): Since $\Phi\colon\mathsf{V}^X\longrightarrow\mathsf{V}$ preserves infima and cotensors, Φ is representable by some $\varphi\in\mathsf{V}^X$, i.e. $\Phi=[\varphi,-]$. Hence, by the lemma above, Φ is a $\mathcal{T}$-graph morphism. We put $\psi\colon=\Phi\cdot\ulcorner a\urcorner$,

$$\begin{tikzcd} {|X|, X^{\mathrm{op}}} \arrow[r, "\ulcorner a\urcorner"] \arrow[dr, "\psi"'] & \mathsf{V}^X \arrow[d, "\Phi"] \\ & \mathsf{V} \end{tikzcd}$$

then $\psi\colon E \,{\circ\!\!\rightharpoondown}\, X$ is a $\mathcal{T}$-distributor by Theorem 1.8. We have, for any $\mathfrak{x}\in TX$ and $x\in X$,

$$\psi(\mathfrak{x})\otimes\varphi(x) = [\varphi,a(\mathfrak{x},-)]\otimes\varphi(x) \le \hom(\varphi(x),a(\mathfrak{x},x))\otimes\varphi(x)\le a(\mathfrak{x},x).$$

On the other hand,

$$\begin{aligned}\bigvee_{\mathfrak{x}\in TX}\psi(\mathfrak{x})\otimes\xi\cdot T\varphi(\mathfrak{x}) &= \bigvee_{\mathfrak{x}\in TX}[\varphi, a(\mathfrak{x},-)]\otimes\xi\cdot T\varphi(\mathfrak{x})\\ &= \bigvee_{\mathfrak{x}\in TX}[\varphi, a(\mathfrak{x},-)\otimes\xi\cdot T\varphi(\mathfrak{x})]\\ &= [\varphi, \bigvee_{\mathfrak{x}\in TX} a(\mathfrak{x},-)\otimes\xi\cdot T\varphi(\mathfrak{x})]\\ &= [\varphi,\varphi]\\ &\geq k,\end{aligned}$$

we have shown that $\psi \dashv \varphi$. □

We conclude that the map $\mathsf{Map}(\mathcal{T}\text{-}\mathsf{Dist})(E,X) \longrightarrow \mathcal{T}\text{-}\mathsf{Frm}(\Omega(X),\mathsf{V}),\ \psi \mapsto [\varphi,-]$ is actually bijective. Finally, for any $x \in X$ and each $\mathcal{T}$-functor $\varphi\colon X \longrightarrow \mathsf{V}$, we have

$$[x^*,\varphi] = \varphi\circ x_* = \bigvee_{\mathfrak{x}\in TX} a(\mathfrak{x},x)\otimes\xi\cdot T\varphi(\mathfrak{x}) = \varphi(x) = \mathrm{ev}_{X,x}(\varphi),$$

which proves the commutativity of the diagram in Theorem 2.13.

3. Examples

We consider first the identity theory for an arbitrary quantale V, cf. Example 1.2 (1); in this case, $\mathcal{T}\text{-}\mathsf{Cat} = \mathsf{V}\text{-}\mathsf{Cat}$ is the category of V-enriched categories. A $\mathcal{T}$-diagram is just an ordinary diagram, and therefore $\mathcal{T}\text{-}\mathsf{Frm}$ is the 2-category having as objects complete (and cocomplete) completely distributive V-categories, and as morphisms all limit- and colimit-preserving functors between them. Writing $\mathsf{V}\text{-}\mathsf{Frm}$ for this category, we *do* have an adjunction

$$\begin{tikzcd}(\mathsf{V}\text{-}\mathsf{Cat})^{\mathrm{op}} \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & \mathsf{V}\text{-}\mathsf{Frm}, \arrow[l, bend right, "\mathrm{pt}"']\end{tikzcd}$$

and $\eta_X\colon X \longrightarrow \mathrm{pt}(\Omega(X))$ is fully faithful for each V-category X. The latter is a consequence of the well-known fact that V is *initially dense* in $\mathsf{V}\text{-}\mathsf{Cat}$ (see [Tho07], for instance), i.e. for each V-category X the source $\mathsf{V}\text{-}\mathsf{Cat}(X,\mathsf{V})$ is initial (jointly fully faithful).

In particular, for $\mathsf{V} = 2$ (the two-element chain), the adjunction above specialises to ordered sets and completely distributive complete lattices

$$\begin{tikzcd}\mathsf{Ord}^{\mathrm{op}} \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & \mathsf{CCD} \arrow[l, bend right, "\mathrm{pt}"']\end{tikzcd}$$

which restricts to a dual equivalence between Ord and the category TAL of totally algebraic complete lattices and suprema and infima preserving maps. And for $\mathsf{V} = [0,\infty]$ (the extended non-negative real numbers) we obtain an adjunction

$$\begin{tikzcd}\mathsf{Met}^{\mathrm{op}} \arrow[r, bend right, "\Omega"'] \arrow[r, phantom, "\perp"] & \mathsf{CDMet} \arrow[l, bend right, "\mathrm{pt}"']\end{tikzcd}$$

where CDMet denotes the category of completely distributive metric spaces and limit- and colimit-preserving contraction maps. This adjunction restricts to a dual equivalence between the full subcategories of *Cauchy complete metric spaces* and totally algebraic metric spaces respectively. Here a metric

space X is completely distributive if it is cocomplete and the left adjoint $S\colon [0,\infty]^{X^{\mathrm{op}}} \longrightarrow X$ of the Yoneda embedding $\mathrm{y}_X\colon X \longrightarrow [0,\infty]^{X^{\mathrm{op}}}$ has a further left adjoint $t_X\colon X \longrightarrow [0,\infty]^{X^{\mathrm{op}}}$. Furthermore, a completely distributive metric space X is totally algebraic if S restricts to an isomorphism $[0,\infty]^{A^{\mathrm{op}}} \cong X$ where $A \hookrightarrow X$ is the equaliser of y_X and t_X. We refer to [Stu07] where complete distributivity and algebraicity are investigated in the context of quantaloid-enriched categories.

Finally, in the remainder of this section we consider the ultrafilter theories of Example 1.2 (2–4); below we denote such a theory as $\mathcal{U}$. As recalled in Example 1.5, if the underlying quantale is $\mathsf{V} = 2$, then $\mathcal{U}_2\text{-}\mathsf{Cat} = \mathsf{Top}$ is the category of topological spaces; and if $\mathsf{V} = [0,\infty]$, then $\mathcal{U}_{[0,\infty]}\text{-}\mathsf{Cat} = \mathsf{App}$ is the category of approach spaces. Our aim is to show how, in general, the notion of $\mathcal{U}$-supremum captures precisely a finiteness condition; so that, consequently, the distributivity law for a $\mathcal{U}$-frame expresses that finite suprema must distribute over arbitrary infima. To prove this, we start with a well-known lemma providing a crucial tool when working with ultrafilters (a proof can be found in [Joh86], for instance):

Lemma 3.1. *Let X be a set, $\mathfrak{j}$ be an ideal and $\mathfrak{f}$ be a filter on X with $\mathfrak{f} \cap \mathfrak{j} = \varnothing$. Then there exists an ultrafilter $\mathfrak{x} \in UX$ with $\mathfrak{f} \subseteq \mathfrak{x}$ and $\mathfrak{x} \cap \mathfrak{j} = \varnothing$.*

For any $\mathcal{U}$-category $X = (X,a)$ and any $A \subseteq X$, the map $\varphi_A\colon X \longrightarrow \mathsf{V}\colon x \mapsto \bigvee\{a(\mathfrak{a},x) \mid \mathfrak{a} \in U(X), A \in \mathfrak{a}\}$ is in fact a $\mathcal{U}$-functor: for it is the composite

$$X \xrightarrow{\ulcorner a \urcorner} \mathsf{V}^{|X|} \longrightarrow \mathsf{V}^{|A|} \xrightarrow{\bigvee} \mathsf{V}$$

of $\mathcal{U}$-functors. Note also that, for $\mathfrak{x} \in UX$ and $A \in \mathfrak{x}$, $\xi \cdot U\varphi_A(\mathfrak{x}) \geq k$. Recall further that U_ξ is the lax extension of the ultrafilter monad to $\mathsf{V}\text{-}\mathsf{Mat}$. In the following proofs we shall write $\ll$ for the totally below relation of V.

Lemma 3.2. *Let $X = (X,a)$ be a $\mathcal{U}$-category. For $\mathfrak{x} \in UX$ and $x \in X$,*

$$\bigwedge\{\varphi_A(x) \mid A \in \mathfrak{x}\} = \bigvee\{U_\xi a(\mathfrak{X},\dot{x}) \mid \mathfrak{X} \in UUX,\ m_X(\mathfrak{X}) = \mathfrak{x}\}.$$

Proof. Clearly, $U_\xi a(\mathfrak{X},\dot{x}) \leq a(\mathfrak{x},x) \leq \hom(\xi \cdot U\varphi_A(\mathfrak{x}),\varphi_A(x)) \leq \varphi_A(x)$ for $\mathfrak{X} \in UUX$ with $m_X(\mathfrak{X}) = \mathfrak{x}$ and $A \in \mathfrak{x}$. Let now $u \in \mathsf{V}$ with $u \ll \bigwedge_{A\in\mathfrak{x}}\varphi_A(x)$. Putting $\mathfrak{j} = \{\mathcal{B} \subseteq UX \mid \forall \mathfrak{y} \in \mathcal{B}\,.\,u \not\leq a(\mathfrak{y},x)\}$ defines an ideal disjoint from $\mathfrak{x}^\# = \{UA \mid A \in \mathfrak{x}\}$. Let $\mathfrak{X} \in UUX$ be an ultrafilter with $\mathfrak{x}^\# \subseteq \mathfrak{X}$ and $\mathfrak{X} \cap \mathfrak{j} = \varnothing$. Then $m_X(\mathfrak{X}) = \mathfrak{x}$ and $U_\xi a(\mathfrak{X},\dot{x}) = \bigwedge_{\mathcal{A}\in\mathfrak{X}}\bigvee_{\mathfrak{a}\in\mathcal{A}} a(\mathfrak{a},x) \geq u$. □

Corollary 3.3. *Let $X = (X,a)$ be a $\mathcal{U}$-category, $\mathfrak{x} \in UX$ and $x \in X$. Then $a(\mathfrak{x},x) = \bigwedge\{\varphi_A(x) \mid A \in \mathfrak{x}\}$.*

Proof. Because $a(\mathfrak{x},x) = \bigvee\{U_\xi a(\mathfrak{X},\dot{x}) \mid \mathfrak{X},\ m_X(\mathfrak{X}) = \mathfrak{x}\} = \bigwedge\{\varphi_A(x) \mid A \in \mathfrak{x}\}$. □

Corollary 3.4. *For each $\mathcal{U}$-category X, the source $\mathcal{U}\text{-}\mathsf{Cat}(X,\mathsf{V})$ is initial (i.e. jointly fully faithful).*

We can now show how the ultrafilter monad allows us to capture a finiteness condition, under some strong assumptions on V:

Proposition 3.5. *Assume that the quantale V satisfies $\top = k$, $\{u \in \mathsf{V} \mid u \ll k\}$ is directed and $k \leq u \vee v$ implies $k \leq u$ or $k \leq v$, for all $u,v \in \mathsf{V}$. Let $\Phi\colon \Omega(X) \longrightarrow \mathsf{V}$ be a V-functor which preserves infima, tensors and cotensors. Then Φ preserves $\mathcal{U}$-suprema if and only if Φ preserves finite suprema.*

Proof. Clearly, if Φ preserves $\mathcal{U}$-suprema then Φ preserves finite suprema. Assume now that Φ preserves finite suprema, we have to show (*) in Proposition 2.15. Note that Φ is necessarily of the form $\Phi = [\varphi,-]$, for some $\varphi \in \Omega(X)$. Furthermore, by our conditions on V and since Φ preserves finite suprema, $\varphi \leq \varphi_1 \vee \varphi_2$ implies $\varphi \leq \varphi_1$ or $\varphi \leq \varphi_2$, for $\varphi_1,\varphi_2 \in \Omega(X)$. We start by showing that there exists an ultrafilter $\mathfrak{x} \in UX$ with $\varphi = a(\mathfrak{x},-)$ and $k = \xi \cdot U\varphi(\mathfrak{x})$. This generalises a well-known property of irreducible closed

subsets of a topological space as well as of approach prime elements in the regular function frame of an approach space (see Proposition 5.7 of [BLVO06]).

To this end, note first that $k = \bigvee\{\varphi(x) \mid x \in X\}$. In fact, with $u = \bigvee\{\varphi(x) \mid x \in X\}$, one has $k = \Phi(\varphi) \leq \Phi(u) = u$. Let now $u \ll k$ and put $A_u = \{x \in X \mid u \leq \varphi(x)\}$. We show that $\varphi \leq \varphi_{A_u}$. Consider the set $A = \{x \in X \mid \varphi(x) \leq \varphi_{A_u}(x)\}$ and put $v = \bigvee\{\varphi(x) \mid x \in X, x \notin A\}$. One has $A_u \subseteq A$, and therefore $k \neq v$ since otherwise there would exist some $x \in X$ with $u \leq \varphi(x)$ and $x \notin A$. Consequently, $\varphi \nleq \varphi \wedge v$. By definition, $\varphi \leq \varphi_{A_u} \vee (\varphi \wedge v)$, and we conclude $\varphi \leq \varphi_{A_u}$. We have shown that the filter base

$$\mathfrak{f} = \{A_u \mid u \ll k\}$$

is disjoint from the ideal

$$\mathfrak{j} = \{B \mid \varphi \nleq \varphi_B\},$$

and therefore Lemma 3.1 provides us with an ultrafilter $\mathfrak{x} \in UX$ with $\mathfrak{x} \cap \mathfrak{j} = \varnothing$. Hence,

$$a(\mathfrak{x}, x) = \bigwedge_{A \in \mathfrak{x}} \varphi_A(x) \geq \varphi(x)$$

Furthermore, for any $u \ll k$,

$$\varphi_{A_u}(x) = \bigvee_{\mathfrak{y} \in UA_u} a(\mathfrak{y}, x) \leq \bigvee_{\mathfrak{y} \in UA_u} \hom(\xi \cdot U\varphi(\mathfrak{y}), \varphi(x)) \leq \hom(u, \varphi(x)),$$

and therefore

$$a(\mathfrak{x}, x) \leq \bigwedge_{u \ll k} \varphi_{A_u}(x) \leq \bigwedge_{u \ll k} \hom(u, \varphi(x)) = \hom(\bigvee_{u \ll k} u, \varphi(x)) = \varphi(x).$$

Let now $\varphi' \in \Omega(X)$. Since $\Phi(\varphi') = [\varphi, \varphi']$, one has $\Phi(\varphi') \otimes \varphi(x) \leq \varphi'(x)$ for every $x \in X$. Finally

$$\Phi(a(\mathfrak{x}, -)) \otimes \xi \cdot U\varphi'(\mathfrak{x}) = \xi \cdot U\varphi'(\mathfrak{x}) = \bigwedge_{A \in \mathfrak{x}} \bigvee_{x \in A} \varphi'(x) \geq \bigwedge_{A \in \mathfrak{x}} \bigvee_{x \in A} \Phi(\varphi') \otimes \varphi(x) \geq \Phi(\varphi') \otimes \xi \cdot U\varphi(\mathfrak{x}) = \Phi(\varphi'). \quad \square$$

As a consequence, in the situation of the proposition above we can modify our definition of $\mathcal{U}$-Frm (see Definition 2.9) by replacing $\mathbb{U}$-algebra with *finite (and hence discrete)* $\mathbb{U}$*-algebra* everywhere; and we do not need the $\mathcal{U}$-graph structure anymore.

In particular, for $\mathsf{V} = \mathsf{2}$ we thus find that the objects of $\mathcal{U}_2$-Frm are complete ordered sets satisfying the co-frame law, and the morphisms are order-preserving maps which preserve all infima and finite suprema. In other words, we arrive at the usual category of co-frames and co-frame homomorphisms. It is well-known (as we recalled in Section 2) that it is involved in a dual adjunction with $\mathcal{U}_2$-$\mathsf{Cat} = \mathsf{Top}$. The situation is similar for $\mathsf{V} = [0, \infty]$. In this case a $\mathcal{U}_{[0,\infty]}$-frame is a complete (that is, one admitting all weighted limits and colimits; not to be confused with Cauchy complete) metric space where "finite colimits commute with arbitrary limits", and a homomorphism is a contraction map preserving all limits and finite colimits. Our perspective differs here from [BLVO06] where so-called "approach frames" were introduced as certain algebras; so far we do not know if both notions are equivalent and therefore leave this as an open problem.

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Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal
E-mail address: `dirk@ua.pt`

Laboratoire de Mathématiques Pures et Appliquées, Université du Littoral-Côte d'Opale, France
E-mail address: `isar.stubbe@lmpa.univ-littoral.fr`